Two-Dimensional Acoustic Field in a Nonuniform Duct Carrying Compressible Flow

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An analytical/numerical linear acoustic solution in a nonuniform two-dimensional duct carrying a compressible mean flow is developed. A quasi-one-dimensional mean flow model is employed together with a consistent expression for the cross-flow velocity. The acoustic solution is obtained using the wave envelope method. Numerical results are compared with those of an existing wave envelope solution which uses a more general mean flow model that includes a thin boundary layer. The results suggest that a thin boundary layer in the flow model may be of minor influence on the sound field in many cases of practical interest.

I. Introduction

SOUND propagation through compressible mean flows in nonuniform ducts has received much attention recently, especially in conjunction with the reduction of noise transmitted out of engine inlets. Generally, analytical work in this area has been confined either to the fully quasi-one-dimensional theory or to propagation in ducts having very slow axial variations in cross-sectional area. For more general multidimensional acoustic propagation there exist a number of studies which are based on various techniques for numerical solution of the governing equations. A comprehensive survey of work which appeared up to the period 1978-79 has been given by Nayfeh[1,2]; more recent studies are cited in Ref. 3.

One of the more promising analytical/numerical techniques for solving the multidimensional linearized acoustic equations is the wave envelope method (WEM) which has been developed by Nayfeh and his co-workers. The fundamental theory associated with this method can be found in Refs. 1 and 2. The bibliography to Ref. 3 provides a guide to later applications and refinements. The current paper reports results of a study of the sound field in a nonuniform, symmetric two-dimensional duct which is based on a modification of the WEM.

The work described in the paper arose in conjunction with a study of the nonlinear behavior of a multidimensional sound field which occurs as it encounters a region where the mean flow approaches sonic conditions. In order to apply asymptotic methods to the nonlinear problem it is necessary to have a detailed description of the analytic behavior of linearized theory in the vicinity of the singularity which it suffers as the flow Mach number approaches unity. As a first step in this study the authors chose to employ a generalized quasi-one-dimensional model for the steady mean flow and to apply the WEM to study the behavior of a two-dimensional sound field in such a flow. The simplification of the mean flow allows all of the coefficients in the equations of the WEM to be expressed in analytical form. As a result, the information necessary as a preliminary step in the analysis of the nonlinear problem is readily available.

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In the course of the work, however, it was found that the linearized acoustic results predicted by the modified WEM were surprisingly accurate when compared to those reported earlier by Nayfeh, who employed a thin boundary-layer flow model. In addition, the simplified flow model leads to substantial reductions in computation time.

It is the purpose of the present paper to discuss the modified WEM analysis of the linearized problem and to present data which compare the results of the analysis to some of those computed using the computer code developed by Nayfeh.[1] As stated previously, the current analysis includes a generalized quasi-one-dimensional model of the mean flowfield. In particular, the pressure, density, and axial velocity of the flow are assumed to be given by the usual one-dimensional theory. The model also, however, accounts for the small, but not negligible, cross-flow velocity component in a manner consistent with the quasi-one-dimensional theory which is valid for a duct whose area varies sufficiently slowly in the axial direction.

A discussion of the steady flow model and the formulation of the resulting two-dimensional linearized acoustic equations is given in Sec. II. Then, in Sec. III the application of the modified WEM to the solution of these equations is outlined although many of the details of the algebraic manipulations are omitted for brevity. The specific analytical coefficients resulting from the simplified flow model are given in the Appendix.

In Sec. IV, numerical results of the present analysis are discussed, and comparisons are made with corresponding results from the more general method. It will be seen that the acoustic pressure magnitudes in the fundamental mode predicted by the present work are in excellent agreement with those from the Nayfeh computer code up to substantially higher Mach numbers and larger boundary-layer thicknesses than were previously thought possible.[1,2] Since no model of the boundary layer is included in the simplified analysis, this indicates that the refractive effect of a thin boundary layer can be of minor importance on the sound field even at relatively high flow Mach number. Instead the local modal reflections caused by the nonuniform mean flow may be largely governed by the cross flow.

Results for multimodal propagation are also presented although they cannot be compared quantitatively with the previous computer code because of the differing duct geometry. However, they are qualitatively similar to the earlier results, again suggesting that modal coupling and reflections depend most strongly on the cross flow and that shear layer effects on the sound may be minor in many cases of practical interest.

II. Formulation and Steady Flow Model

For simplicity, the current work is confined to the case of a two-dimensional, symmetric duct with rigid walls at $y = \pm d(x)$ as indicated in Fig. 1, although extension to rectangular and circular ducts is straightforward. The equations expressing conservation of mass, linear momentum, and energy for an inviscid ideal gas are written in the form

$$c_s\bar{\rho}_t + (\bar{\rho}\bar{u})_x + (\bar{\rho}\bar{v})_y = 0$$

$$\bar{\rho}(c_s\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y) + \bar{p}_x = 0$$

$$\bar{\rho}(c_s\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y) + \bar{p}_y = 0$$

$$c_s\bar{p}_t + \bar{u}\bar{p}_x + \bar{v}\bar{p}_y = \frac{\gamma\bar{p}}{\bar{\rho}}(c_s\bar{\rho}_t + \bar{u}\bar{\rho}_x + \bar{v}\bar{\rho}_y) \tag{1}$$

in which $\bar{\rho}$, $\bar{u}$, $\bar{v}$, and $\bar{p}$ are the dimensional total fluid density, axial velocity, cross-flow velocity, and pressure, respectively; γ is the specific heat ratio of the gas and c_s is the stagnation value of sound speed in the gas. The dimensionless independent space variables x and y are measured in units of d^*, while the dimensionless time t is in units of d^*/c_s, where d^* is the half-width of the duct at its throat. The boundary condition at the rigid duct walls is expressible as $\bar{v} = \pm d'(x)\bar{u}$ on $y = \pm d(x)$, where $d(0) = 1$.

The total fluid quantities are decomposed into steady and unsteady components according to

$$\bar{\rho}(x,y,t) = R(x,y) + \bar{\rho}_1(x,y,t)$$

$$\bar{u}(x,y,t) = U(x,y) + \bar{u}_1(x,y,t)$$

$$\bar{v}(x,y,t) = V(x,y) + \bar{v}_1(x,y,t)$$

$$\bar{p}(x,y,t) = P(x,y) + \bar{p}_1(x,y,t) \tag{2}$$

Equations (2) are substituted into Eqs. (1) assuming that R, U, V, and P satisfy the steady version of Eqs. (1). Then, if all but first-order terms in the small unsteady quantities are neglected, Eqs. (1) reduce to the usual linearized acoustic equations as follows

$$c_s\bar{\rho}_{1_t} + (R\bar{u}_1 + U\bar{\rho}_1)_x + (R\bar{v}_1 + V\bar{\rho}_1)_y = 0$$

$$c_sR\bar{u}_{1_t} + (R\bar{u}_1 + U\bar{\rho}_1)U_x + RU\bar{u}_{1_x} + (R\bar{v}_1 + V\bar{\rho}_1)U_y + RV\bar{u}_{1_y} + \bar{p}_{1_x} = 0$$

$$c_sR\bar{v}_{1_t} + (R\bar{u}_1 + U\bar{\rho}_1)V_x + RU\bar{v}_{1_x} + (R\bar{v}_1 + V\bar{\rho}_1)V_y + RV\bar{v}_{1_y} + \bar{p}_{1_y} = 0$$

$$c_sR\bar{p}_{1_t} + (U\bar{\rho}_1 + R\bar{u}_1)P_x + RU\bar{p}_{1_x} + (R\bar{v}_1 + V\bar{\rho}_1)P_y + RV\bar{p}_{1_y} = \gamma[c_sP\bar{\rho}_{1_t} + (P\bar{u}_1 + U\bar{p}_1)R_x + PU\bar{\rho}_{1_x} + (P\bar{v}_1 + V\bar{p}_1)R_y + PV\bar{\rho}_{1_y}] \tag{3}$$

Further, the wall boundary condition on the acoustic field becomes

$$\bar{v}_1 = \pm d'(x)\bar{u}_1 \quad \text{on} \quad y = \pm d(x) \tag{3a}$$

In order to make progress in analyzing Eqs. (3) it is necessary to know R, U, V, and P, which have been assumed to satisfy the steady form of Eqs. (1). These quantities, in general, cannot be obtained in analytical form as is well known. Since the mean flow quantities appear only as coefficients in Eqs. (3), it has become common to employ approximate models of the flow. In the work of Refs. 1-3, for example, a combination of a quasi-one-dimensional core flow and a thin boundary layer was used in an attempt to model the effects on the sound field both of spatial gradients and of a viscous boundary layer in the mean flow. In the present work a simpler approximate flow model will be used which is based on quasi-one-dimensional theory but also accounts consistently for a small but non-negligible cross-flow velocity component. The mean flow approximation requires that the duct cross-sectional area varies sufficiently slowly in the axial direction and it does not include any simulation of a viscous boundary layer.

It can be shown by a formal asymptotic analysis in terms of a small parameter ϵ, which can be a measure, for example, of the maximum duct wall slope, that the basic steady flow quantities can be expressed in series form as

$$R(x,y) = R_0 + \epsilon^2R_2 + \ldots$$

$$P(x,y) = P_0 + \epsilon^2P_2 + \ldots$$

$$U(x,y) = U_0 + \epsilon^2U_2 + \ldots$$

$$V(x,y) = \epsilon V_1 + \epsilon^3V_3 + \ldots \tag{4}$$

When these expansions are substituted into the steady version of Eqs. (1) it is found that R_0, P_0, and U_0 are independent of y and satisfy the well-known quasi-one-dimensional steady flow equations

$$(R_0U_0A)' = 0, \quad R_0U_0U_0' + P_0' = 0, \quad P_0' - c_0^2R_0' = 0 \tag{5}$$

in which $A(x)$ is the duct cross-sectional area, $c_0(x)$ is the speed of sound given by $c_0^2 = \gamma P_0/R_0$, and the primes denote differentiation with respect to the single independent variable x. In addition, the formal analysis leads to the result that satisfaction of the steady continuity equation at the leading order requires

$$\epsilon\frac{\partial V_1(x,y)}{\partial y} = -\frac{1}{R_0(x)}\frac{\mathrm{d}}{\mathrm{d}x}[R_0(x)U_0(x)] \tag{6}$$

It is emphasized here that if the flow were approximated by keeping only $\mathcal{O}(1)$ terms in expansions (4), then $V(x,y)$ would vanish. This is inconsistent, however, for two reasons. First, the wall boundary condition Eq. (3a) would not be satisfied. Second, as Eq. (6) indicates, while the leading order approximation for V is only $\mathcal{O}(\epsilon)$ in magnitude, it is nevertheless as large as $(R_0U_0)'$ which has been retained in Eqs. (5). The leading approximation for V, determined from Eq. (6), must be retained in the flow model if the steady continuity equation and the wall tangency condition are to be satisfied consistently.

Thus, for the purposes of the present work the mean flow is assumed to be given by

$$R = R_0(x), \quad P = P_0(x), \quad U = U_0(x), \quad V = U_0(x)\beta(x,y)$$

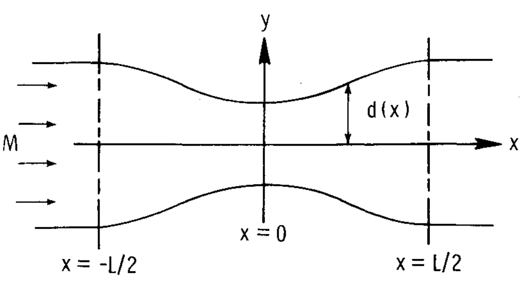

Fig. 1 Typical duct geometry.

where

$$\beta(x,y) = -\frac{(R_0 U_0)'}{R_0 U_0} y \tag{7}$$

which is obtained by integration of Eq. (6) taking account of the symmetry of the flow about $y=0$. Of course, R_0, U_0, P_0, and thus, β are known analytically in terms of the axial flow Mach number after solving the usual one-dimensional system (5).

Finally, the system of Eqs. (3) is cast into a more convenient form by defining dimensionless acoustic quantities according to

$$\bar{\rho}_1(x,y,t) = R_0(x)\rho(x,y,t) \qquad \bar{p}_1(x,y,t) = P_0(x)p(x,y,t)$$

$$\bar{u}_1(x,y,t) = U_0(x)u(x,y,t) \qquad \bar{v}_1(x,y,t) = U_0(x)v(x,y,t) \tag{8}$$

Substitution of the definitions in Eqs. (8) into the system (3) and use of the simplified mean flow model above yields the following system of acoustic equations of motion:

$$\sqrt{G}\rho_t + M(\rho+u)_x + Mv_y + M\beta\rho_y + \frac{M'}{G}(1-M^2)u = 0$$

$$\sqrt{G}u_t + Mu_x + \frac{M'}{G}(2u+\rho-p) + \frac{p_x}{\gamma M} + M\beta u_y = 0$$

$$\sqrt{G}v_t + Mv_x + \frac{M'}{G}M^2 v + \frac{1}{\gamma M}p_y + \left(M\beta_x + \frac{\beta M'}{G}\right)(\rho+u)$$

$$+ M\beta v_y - \frac{\beta M'}{G}(1-M^2)\rho = 0$$

$$\sqrt{G}(p-\gamma\rho)_t + M(p-\gamma\rho)_x + M\beta(p-\gamma\rho)_y = 0 \tag{9}$$

In Eqs. (9) $M(x)$ is the steady flow Mach number $U_0(x)/c_0(x)$, and

$$G(x) = c_s^2/c_0^2(x) = 1 + \frac{\gamma-1}{2}M^2(x)$$

Also, in deriving Eqs. (9) use has been made of the fact that Eq. (7) and Eqs. (5) imply[5]

$$\beta_y = A'/A = -M'(1-M^2)/MG$$

The wall boundary condition Eq. (3a) becomes, now,

$$v(x,y,t) = \pm d'(x)u(x,y,t) \quad \text{on } y = \pm d(x) \tag{9a}$$

The system of Eqs. (9) describes two-dimensional sound propagation through a generalized quasi-one-dimensional flow which includes the effects of a small, but not negligible, cross-flow velocity component. Approximate harmonic solutions to Eqs. (9) subject to the boundary condition Eq. (9a) and to suitable source and exit conditions will be derived in the next section using the wave envelope method.

III. Solution of Acoustic Equations

In order to apply the wave envelope method[1-3] to system (9), approximate harmonic solutions are sought in the form

$$\rho = \sum_{n=0}^{N-1} [\psi_n^\rho(x,y)E_n(x) + \tilde{\psi}_n^\rho(x,y)\tilde{E}_n(x)]e^{-i\omega t}$$

$$p = \sum_{n=0}^{N-1} [\psi_n^p(x,y)E_n(x) + \tilde{\psi}_n^p(x,y)\tilde{E}_n(x)]e^{-i\omega t}$$

$$u = \sum_{n=0}^{N-1} [\psi_n^u(x,y)E_n(x) + \tilde{\psi}_n^u(x,y)\tilde{E}_n(x)]e^{-i\omega t}$$

$$v = \sum_{n=0}^{N-1} [\psi_n^v(x,y)E_n(x) + \tilde{\psi}_n^v(x,y)\tilde{E}_n(x)]e^{-i\omega t} \tag{10}$$

in which

$$E_n(x) = S_n(x)\exp[i\int k_n(x)\,dx]$$

$$\tilde{E}_n(x) = \tilde{S}_n(x)\exp[i\int \tilde{k}_n(x)\,dx] \tag{11}$$

In these equations ω is a dimensionless frequency measured in units of c_s/d^*, the various $\psi_n(x,y)$ are the quasiparallel mode shapes, the $k_n(x)$ are the quasiparallel local wave numbers corresponding to ψ_n, and the $S_n(x)$ are complex amplitudes which are to be determined. In the present work the tilde refers to downstream propagation, and each acoustic quantity is expressed in terms of an upstream (no tilde) and a downstream propagating component. The summations in Eqs. (10) are over a finite number N of modes, $n=0$ representing the fundamental.

As discussed in detail in Refs. 1-3, the form of solution (10) is predicated on the assumption that rapid axial oscillations are described by the exponential factors in E_n and $\tilde{E}_n$, while the unknown complex amplitude factors S_n and $\tilde{S}_n$ represent slowly varying amplitude and phase corrections necessitated by the varying duct geometry and mean flow.

The quasiparallel mode shapes and wave numbers are found by seeking solutions of the form $\psi \exp i(kx-\omega t)$ to the uniform area version of the system of Eqs. (9) and (9a) using the local Mach number and duct width. Thus, they satisfy

$$-i\kappa\psi^\rho + ikM\psi^u + M\psi_y^v = 0 \qquad -i\kappa\psi^u + \frac{ik}{\gamma M}\psi^p = 0$$

$$-i\kappa\psi^v + \frac{1}{\gamma M}\psi_y^p = 0 \qquad \psi^p - \gamma\psi^\rho = 0 \tag{12}$$

with $\psi^v = 0$ at $y = \pm d$, where $\kappa = \omega\sqrt{G} - kM$. System (12) is easily reduced into the standard second-order differential equation for ψ^p corresponding to a locally uniform flow at Mach number $M(x)$. For the rigid wall boundary condition the resulting eigenvalue problem can be solved for the quasiparallel modes as follows:

Upstream propagation

$$\psi_n^\rho = \cos\lambda_n(d-y); \qquad \psi_n^p = \gamma\cos\lambda_n(d-y)$$

$$\psi_n^u = \frac{k_n}{\kappa_n M}\cos\lambda_n(d-y); \quad \psi_n^v = \frac{-i\lambda_n}{\kappa_n M}\sin\lambda_n(d-y) \tag{13}$$

where

$$k_n = \frac{-\omega M\sqrt{G} - \sqrt{\omega^2 G - (1-M^2)\lambda_n^2}}{1-M^2} \quad \text{and} \quad \kappa_n = \omega\sqrt{G} - k_n M$$

Downstream propagation

$$\tilde{\psi}_n^\rho = \cos\lambda_n(d-y); \quad \tilde{\psi}_n^p = \gamma\cos\lambda_n(d-y)$$

$$\tilde{\psi}_n^u = \frac{\tilde{k}_n}{\tilde{\kappa}_n M}\cos\lambda_n(d-y); \quad \tilde{\psi}_n^v = \frac{-i\lambda_n}{\tilde{\kappa}_n M}\sin\lambda_n(d-y) \tag{14}$$

where

$$\tilde{k}_n = \frac{-\omega M\sqrt{G} + \sqrt{\omega^2 G - (1-M^2)\lambda_n^2}}{1-M^2} \quad \text{and} \quad \tilde{\kappa}_n = \omega\sqrt{G} - \tilde{k}_n M$$

The eigenvalue λ_n in Eqs. (13) and (14) is given by

$$\lambda_n = n\pi/2d(x)$$

Now, because approximation (10) contains a specified y dependence, it cannot satisfy the system (9) and (9a) exactly. Instead, the expressions (10) must be subjected to constraints which define the sense in which they are to approximate the acoustic quantities. The constraint used in the original development of the WEM was to require "the deviations from the quasiparallel solution to be orthogonal to every solution of the adjoint quasiparallel problem."[1] A justification of this requirement can be made using the theory of solvability conditions for inhomogeneous differential operators. It has been found to yield accurate approximations when compared to acoustic solutions derived by alternate numerical analyses.[1,3]

In order to apply the contraint, the adjoint quasiparallel modes must be determined. These can be found explicitly in the present case by the standard procedure of multiplying each equation in system (12) by corresponding adjoint functions ϕ_i, $i=1,\ldots,4$, adding the results, and integrating over y from $-d$ to d. Integrations by parts and use of the boundary condition $\psi^v=0$ at $y=\pm d$ result in identification of the upstream propagating adjoint functions as

$$\phi_{1n} = \cos\lambda_n(d-y); \qquad \phi_{2n} = \frac{k_n M}{\kappa_n}\cos\lambda_n(d-y)$$

$$\phi_{3n} = \frac{iM}{\kappa_n}\lambda_n \sin\lambda_n(d-y); \quad \phi_{4n} = \frac{1}{\gamma}\cos\lambda_n(d-y) \tag{15}$$

with corresponding downstream forms identical to Eqs. (15) but having k_n and κ_n replaced by $\tilde{k}_n$ and $\tilde{\kappa}_n$, respectively.

Once the adjoint functions are determined, the $2N$ compatibility conditions are derived formally as in Ref. 1 by multiplying each equation in system (9) by its corresponding ϕ and integrating over $-d \le y \le d$

$$\int_{-d}^{d}\Big[-i\omega\sqrt{G}\rho + M(u+\rho)_x + Mv_y + M\beta\rho_y + \frac{M'}{G}(1-M^2)u\Big]\Phi_{1m}\mathrm{d}y + \int_{-d}^{d}\Big[-i\omega\sqrt{G}u + Mu_x + \frac{M'}{G}(2u+\rho-p) + \frac{p_x}{\gamma M} + M\beta u_y\Big]\Phi_{2m}\mathrm{d}y$$
$$+\int_{-d}^{d}\Big[-i\omega\sqrt{G}v + Mv_x + \frac{M'M^2}{G}v + \frac{p_y}{\gamma M} + \Big(M\beta_x + \frac{\beta M'}{G}\Big)(\rho+u) + M\beta v_y - \frac{M'\beta}{G}(1-M^2)\rho\Big]\Phi_{3m}\mathrm{d}y + \int_{-d}^{d}[-i\omega\sqrt{G}(p-\gamma\rho)$$
$$+M(p-\gamma\rho)_x + M\beta(p-\gamma\rho)_y]\Phi_{4m}\mathrm{d}y = 0 \tag{16}$$

where Φ_{im}, $i=1,\ldots,4$, stands for each of ϕ_{im} and $\tilde{\phi}_{im}$.

Finally, the assumed solution Eq. (10) is substituted into Eq. (16) making use of Eqs. (11) and (13-15). Each of the integrations over y can be performed explicitly in the present case, and there results a system of $2N$ first-order ordinary differential equations which determine S_n and $\tilde{S}_n$ as follows.

$$\sum_{n=0}^{N-1} Q_{mn}\exp(i\textstyle\int k_n \mathrm{d}x)\dfrac{\mathrm{d}S_n}{\mathrm{d}x} + \tilde{Q}_{mn}\exp(i\textstyle\int \tilde{k}_n \mathrm{d}x)\dfrac{\mathrm{d}\tilde{S}_n}{\mathrm{d}x}$$
$$+F_{mn}\exp(i\textstyle\int k_n\mathrm{d}x)S_n + \tilde{F}_{mn}\exp(i\textstyle\int\tilde{k}_n\mathrm{d}x)\tilde{S}_n = 0$$

$$\sum_{n=0}^{N-1} T_{mn}\exp(i\textstyle\int k_n \mathrm{d}x)\dfrac{\mathrm{d}S_n}{\mathrm{d}x} + \tilde{T}_{mn}\exp(i\textstyle\int \tilde{k}_n \mathrm{d}x)\dfrac{\mathrm{d}\tilde{S}_n}{\mathrm{d}x}$$
$$+G_{mn}\exp(i\textstyle\int k_n\mathrm{d}x)S_n + \tilde{G}_{mn}\exp(i\textstyle\int\tilde{k}_n\mathrm{d}x)\tilde{S}_n = 0 \tag{17}$$

Each of the coefficients in Eqs. (17) is known explicitly as a function of $M(x)$, ω, and the duct geometry $d(x)$. Their specific forms, which follow after considerable algebra, are given in the Appendix.

One further theoretical point is of interest before proceding to a discussion of numerical solutions to system (17). Because the flow model used in the current analysis is quasi-one-dimensional, there naturally arises a question as to the relationship between the solution of Eq. (10) for a single plane mode and the solution predicted by a completely quasi-one-dimensional acoustic theory. If one takes $N=1$ in Eqs. (10), then they become (omitting the exponential time factor)

$$\rho = S_0(x)\exp\Big(-i\int\frac{\omega\sqrt{G}}{1-M}\mathrm{d}x\Big) + \tilde{S}_0(x)\exp\Big(i\int\frac{\omega\sqrt{G}}{1+M}\mathrm{d}x\Big)$$

$$u = \frac{-S_0(x)}{M}\exp\Big(-i\int\frac{\omega\sqrt{G}}{1-M}\mathrm{d}x\Big) + \frac{\tilde{S}_0(x)}{M}\exp\Big(i\int\frac{\omega\sqrt{G}}{1+M}\mathrm{d}x\Big)$$

$$v=0, \qquad p=\gamma\rho \tag{18}$$

if Eqs. (13) and (14) are used. Within a fully quasi-one-dimensional acoustic theory the equations of motion, assuming $p=\gamma\rho$, are known to be[4]

$$-i\omega\sqrt{G}\rho + M(\rho+u)_x = 0$$

$$-i\omega\sqrt{G}u + Mu_x + \frac{1}{M}\rho_x + \frac{M'}{G}[2u-(\gamma-1)\rho] = 0 \tag{19}$$

in the notation of the current paper. If solutions of the form of Eqs. (18) are substituted into Eqs. (19) it is found that this system is satisfied exactly provided that

$$\frac{\mathrm{d}S_0}{\mathrm{d}x} = \frac{M'}{2}\Big[\frac{1}{M} + \frac{(\gamma-1)M+2}{(1-M)G}\Big]S_0 + \frac{M'}{2}\Big[-\frac{1}{M} + \frac{(\gamma-1)M-2}{(1-M)G}\Big]\tilde{S}_0\exp\Big(i\int\frac{2\omega\sqrt{G}}{1-M^2}\mathrm{d}x\Big)$$

$$\frac{\mathrm{d}\tilde{S}_0}{\mathrm{d}x} = \frac{M'}{2}\Big[-\frac{1}{M} + \frac{(\gamma-1)M+2}{(1+M)G}\Big]S_0\exp\Big(-i\int\frac{2\omega\sqrt{G}}{1-M^2}\mathrm{d}x\Big) + \frac{M'}{2}\Big[\frac{1}{M} + \frac{(\gamma-1)M-2}{(1+M)G}\Big]\tilde{S}_0$$

These two equations are identical to those obtained from Eqs. (17) for $N=1$, i.e., for the case in which only the plane mode is included in the WEM. This leads to the conclusion that the plane mode in the modified WEM propagates according to purely quasi-one-dimensional acoustic theory. This fact was used in the current work as a partial check on the accuracy of numerical computations by making use of alternative methods of solution of the quasi-one-dimensional system, Eq. (19).

In the following section, numerical results illustrating the predictions of the modified WEM analysis will be discussed. They are obtained by integration of the system of Eqs. (17) using a fourth-order Runge-Kutta scheme. $2N$ linearly independent solutions are generated by successively setting all but one of the S_n and $\tilde{S}_n$ equal to zero at a source location and integrating Eqs. (17) over the length of the duct to an exit location. Source and exit boundary conditions are then satisfied by linear combinations of these solutions. The procedure is essentially the same as outlined in Ref. 1.

IV. Numerical Results and Discussion

For the purposes of illustrating the two-dimensional theory developed in the current paper, results are calculated for a quartic duct shape

$$d(x)=1+8\left(\frac{x}{L}\right)^2-16\left(\frac{x}{L}\right)^4, \quad |x|<L/2$$

interposed between two semi-infinite uniform sections lying in $|x|>L/2$ (see Fig. 1). The source condition in each case to be shown is a given incident wave of unit amplitude in the plane mode, at $x=L/2$ for upstream propagation or at $x=-L/2$ for downstream propagation. At the opposite end in each case ($x=\mp L/2$, respectively) a no reflection condition is specified, i.e., the amplitudes of all incoming waves are set equal to zero. It is noted that because of symmetry in these cases the plane incident wave excites only symmetric modes (n even). Thus, on the figures referring to multimodal propagation, the notation N_s is used to denote the number of symmetric modes included in the summations of Eqs. (10).

Comparison of numerical results will be made with those calculated using the computer code developed by Nayfeh and his co-workers.[1] In that work a more realistic flow model was used which consists of a quasi-one-dimensional core flow fitted with a thin boundary layer at the duct walls. The previous work treated ducts of circular cross section, so that quantitative comparisons are restricted to cases in which only the fundamental mode is considered.

Figures 2-4 illustrate results calculated for $\omega=1$, a frequency at which only the plane mode propagates. The modified WEM in this case reproduces quasi-one-dimensional acoustic theory, as discussed in Sec. III. Excluding higher order modes allows some assessment of the limitations of the simplified flow model by comparison of the results with those calculated using the method of Ref. 1. The absence of intermodal coupling ensures that differences between the two predictions are entirely attributable to boundary-layer effects.

On Fig. 2 is shown the axial variation of the acoustic pressure magnitude when the throat Mach number is 0.3, and a plane wave is incident in the downstream direction. The prediction of the modified WEM is indicated by the solid line, and the two sets of symbols denote centerline calculations from the Nayfeh computer code using constant boundary-layer displacement thicknesses $\delta=0.001$ and $\delta=0.01$. It is seen that the results are virtually identical. A corresponding calculation for $M(0)=0.5$ appears in Fig. 3, where again the effect of the boundary layer on the centerline pressure is seen to be negligible for $\delta=0.001$ and nearly so for $\delta=0.01$. Figure 4 indicates that including the layer in the flow model can affect the field in and upstream of the throat region at high throat Mach number. However, there is still excellent agreement between the two predictions for the case $\delta=0.001$.

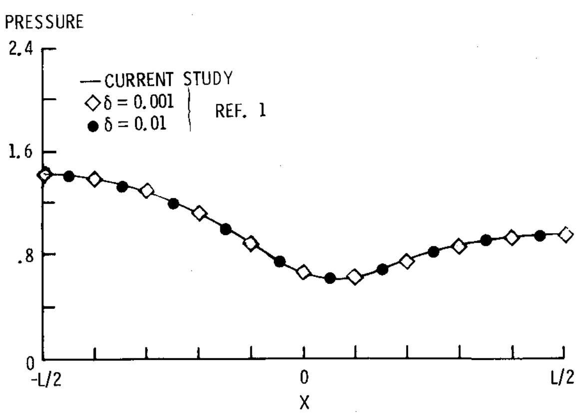

Fig. 2 Pressure magnitude vs axial distance for downstream propagation: $L=1$, $\omega=1$, $M(0)=0.3$, $N_s=1$.

Calculations using the program of Ref. 1 were also made for the pressure magnitude along the duct wall in the preceding cases. Both the 0.3 and 0.5 throat Mach numbers resulted in pressure distributions identical to those of Figs. 2 and 3, indicating no refractive effect of the boundary layer. The same conclusion applies to the case $M(0)=0.85$ for the thinner layer. Displacement thicknesses of the order of 1% do give rise to more significant refraction at high Mach number. However, for $M(0)$ above about 0.8 it is known that linearized theory is inadequate to describe the sound field, especially for upstream propagation, because of the growing influence of the near-sonic singularity.[4]

These results, and others, indicate that the presence of a shear layer in the mean flow may be of much less influence on the propagation of the fundamental mode than was previously concluded,[1,2] at least for situations in which linear theory provides a correct description of the sound field. The continuous reflections which occur in the nonuniform flow appear to be adequately described if the shear layer is ignored provided that the cross flow is properly taken into account. Since significant refractive effects at high Mach number, even for $\delta=0.001$, were attributed to the shear layer in Refs. 1 and 2, it is appropriate to point out that the suppression of two-dimensional effects used there to reproduce a nearly one-dimensional flow included setting the radial velocity to zero. As discussed in Sec. II, this appears to be inconsistent with the steady flow equations of motion.

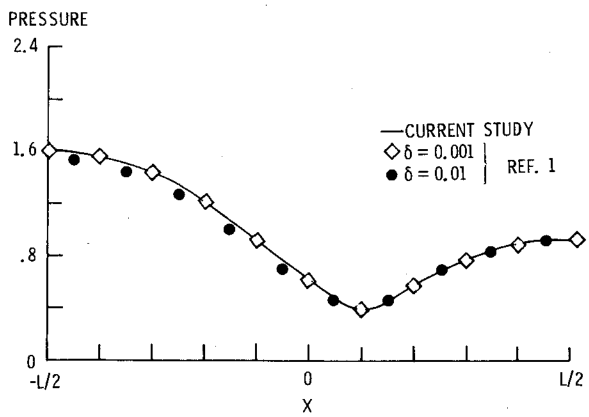

Fig. 3 Pressure magnitude vs axial distance for downstram propagation: $L=1$, $\omega=1$, $M(0)=0.5$, $N_s=1$.

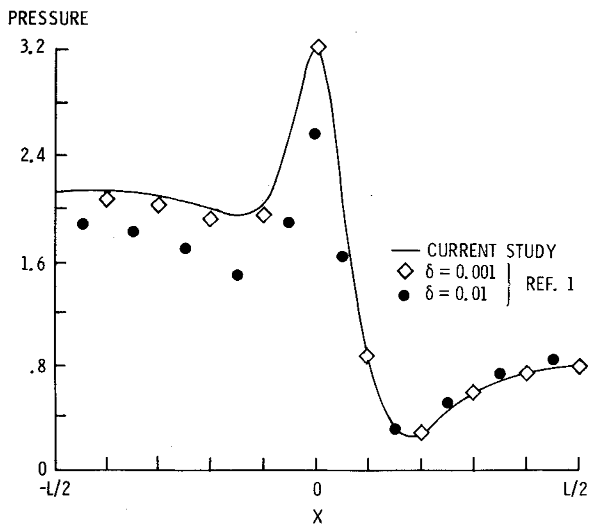

Fig. 4 Pressure magnitude vs axial distance for downstream propagation: $L=1$, $\omega=1$, $M(0)=0.85$, $N_s=1$.

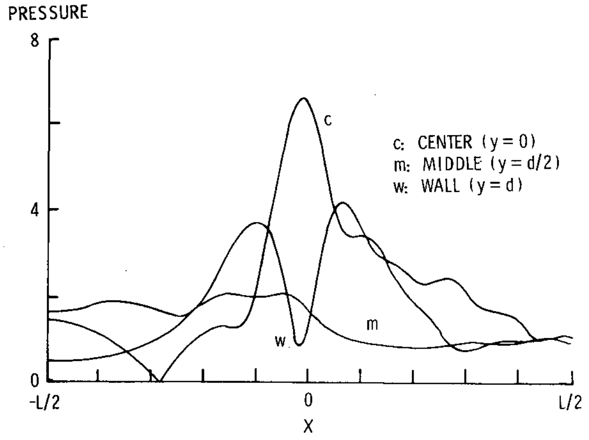

Fig. 5 Pressure magnitude vs axial distance for upstream propagation: $L=4$, $\omega=7.5$, $M(0)=0.6$, $N_s=4$.

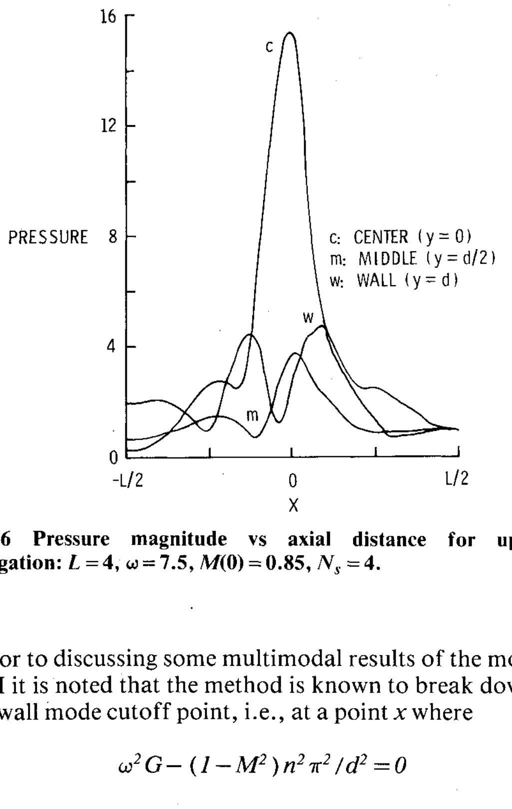

Fig. 6 Pressure magnitude vs axial distance for upstream propagation: $L=4$, $\omega=7.5$, $M(0)=0.85$, $N_s=4$.

Prior to discussing some multimodal results of the modified WEM it is noted that the method is known to break down at a hard wall mode cutoff point, i.e., at a point x where

$$\omega^2 G - (1-M^2)n^2\pi^2/d^2 = 0$$

for some n (see Eqs. 13 and 14). The numerical integration of Eqs. (17) cannot be carried through such a point because the coefficients Q_{mm} and $\tilde{T}_{mm}$ vanish. As a result, the summations in the approximate solutions must be restricted to propagating modes. In uniform ducts this appears to yield good agreement in cases where comparisons can be made with alternate solution techniques,[1,2] and, in fact, Fig. 11 of Ref. 1 indicates that the approximation may be quite accurate even if one or two of the highest order propagating modes are omitted. For nonuniform ducts, however, the situation is more complex, primarily because a mode may go through cutoff at several points along the duct if the Mach number and frequency are sufficiently high. For example, in the duct of Fig. 1, if $M(0)=0.85$ and $\omega=7.5$ then five symmetric modes propagate in each direction at $x=\pm L/2$ and near $x=0$, but the fifth is cut off over two ranges of x symmetrically located on either side of the throat. In the present application of the WEM to multimodal propagation this mode is ignored and only the four which propagate over the entire length of the duct are included. In every case to be shown, however, at most one such mode is ignored.

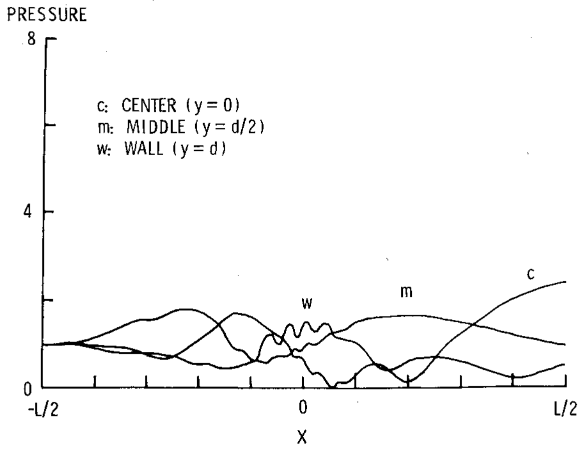

Fig. 7 Pressure magnitude vs axial distance for downstream propagation: $L=4$, $\omega=7.5$, $M(0)=0.85$, $N_s=4$.

Figures 5-7 depict results calculated for the two-dimensional duct at $\omega=7.5$. In each case, four symmetric modes propagate in each direction. The behavior of the pressure magnitude for $M(0)=0.6$ shown in Fig. 5 is typical for upstream propagation against moderate throat Mach numbers. There is a strong intermodal coupling in the nonuniform section which gives rise to complicated interference effects, the net result being a major refraction of sound toward the duct centerline near the throat. This occurs even though no boundary layer is included in the flow model, and it is qualitatively similar to the behavior calculated for rigid circular ducts in Ref. 3. Again, the results suggest that the modal reflections and interactions are governed mainly by the gradients in the axial flow and by the cross-flow velocity, and that actual refraction by a thin shear layer may well be of minor consequence.

A similar result appears in Fig. 6 for $M(0)=0.85$. The pressure is refracted strongly toward the centerline near $x=0$, and the developing high-subsonic singularity yields a large centerline pressure in this region. It is noted, with reference to both Figs. 5 and 6, that because the distribution of pressure in the nonuniform section is a result of mode interference the actual transverse variation in pressure is very sensitive to both frequency and Mach number. Thus, at $x=-L/2$ in both cases the pressure at the wall is higher than that at the centerline, but the pressure difference between the two points is nearly ten times as large at the higher Mach number. At other frequencies it has been found that the centerline pressure can remain above the wall value over the entire upstream half of the duct.

Finally, Fig. 7 is included as as example of downstream propagation. In this case, the sound is refracted toward the wall in the throat region, as would be expected, but at $x=L/2$ the pressure magnitude at the wall has dropped well below that at the centerline. It is also of interest that the pressure near $x=0$ remains small, indicating that the near-sonic singularity does not develop for downstream propagation until the throat Mach number becomes much higher than 0.85.

Appendix

In this Appendix the explicit expressions for the coefficients in Eqs. (17) are given. For simplicity, the following definitions are used

$$I_{mn}=2d(x),\ K_{mn}=J_{mn}=\bar{J}_{mn}=0 \text{ for } n=m=0$$

$$I_{mn}=K_{mn}=d(x),\ J_{mn}=\bar{J}_{mn}=\frac{d^2(x)}{n\pi} \text{ for } n=m\neq 0$$

$$I_{mn} = K_{mn} = 0 \quad \text{for } n \neq m$$

$$J_{mn} = \frac{2nd^2(x)}{\pi(n^2 - m^2)} [1 + (-1)^{n+m}] \quad \text{for } n \neq m$$

$$\bar{J}_{mn} = \frac{-2md^2(x)}{\pi(n^2 - m^2)} [1 + (-1)^{n+m}] \quad \text{for } n \neq m \tag{A1}$$

Further, the following notation is introduced:

$$A_{mn} = M + \frac{k_n}{\kappa_n}\left(1 + M\frac{k_m}{\kappa_m}\right), \quad \tilde{A}_{mn} = M + \frac{\tilde{k}_n}{\tilde{\kappa}_n}\left(1 + M\frac{k_m}{\kappa_m}\right)$$

$$B_{mn} = M + \frac{k_n}{\kappa_n}\left(1 + M\frac{\tilde{k}_m}{\tilde{\kappa}_m}\right), \quad \tilde{B}_{mn} = M + \frac{\tilde{k}_n}{\tilde{\kappa}_n}\left(1 + M\frac{\tilde{k}_m}{\tilde{\kappa}_m}\right)$$

$$C_n = \frac{k_n}{\kappa_n}, \quad \tilde{C}_n = \frac{\tilde{k}_n}{\tilde{\kappa}_n}, \quad D_n = \frac{\lambda_n}{\kappa_n}, \quad \tilde{D}_n = \frac{\lambda_n}{\tilde{\kappa}_n} \tag{A2}$$

in which k_n, κ_n, and λ_n are defined in Eqs. (13) and (14). Two other quantities are defined by

$$\alpha_1 = \frac{(1-M^2)M''}{G} - \left(\frac{M'}{G}\right)^2 M\left[\frac{3(\gamma+1)}{2} - \frac{(\gamma-1)M^2}{2}\right]$$

$$\alpha_2 = \frac{(1-M^2)M''}{G} - \frac{M'}{MG}\left[1 + \frac{3(\gamma-1)M^2}{2} - \frac{(\gamma-3)M^4}{2}\right] \tag{A3}$$

Then, using Eqs. (A1-A3), the coefficients of the system of Eqs. (17) can be written as

$$Q_{mn} = (A_{mn} + C_m)I_{mn} + MD_nD_mK_{mn}$$

$$\tilde{Q}_{mn} = (\tilde{A}_{mn} + C_m)I_{mn} + M\tilde{D}_nD_mK_{mn} \equiv 0$$

$$T_{mn} = (B_{mn} + \tilde{C}_m)I_{mn} + MD_n\tilde{D}_mK_{mn} \equiv 0$$

$$\tilde{T}_{mn} = (\tilde{B}_{mn} + \tilde{C}_m)I_{mn} + M\tilde{D}_n\tilde{D}_mK_{mn}$$

$$F_{mn} = i[(\lambda_mD_m - \lambda_nD_n)I_{mn} + (\lambda_mD_n - \lambda_nD_m)K_{mn} - D_m(\alpha_1C_n + \alpha_2M)\bar{J}_{mn}] + \left[-A_{mn}\frac{d'}{d} + M\left(\frac{C_n}{M}\right)' \times (1 + MC_m) + \frac{M'}{G}(2C_mC_n + M(1-\gamma)C_m)\right]I_{mn} + D_m\left[M^2\left(\frac{D_n}{M}\right)' + D_n\frac{M'}{G}\right]K_{mn} + [\lambda_n'(A_{mn} + C_m) - MD_nD_m\lambda_m']J_{mn} + [\lambda_m'(A_{mn} - C_n) - MD_nD_m\lambda_n'] \times \bar{J}_{mn} + d'A_{mn}[1 + (-1)^{n+m}]$$

$$\tilde{F}_{mn} = i[(\lambda_mD_m - \lambda_n\tilde{D}_n)I_{mn} + (\lambda_m\tilde{D}_n - \lambda_nD_m)K_{mn} - D_m(\alpha_1\tilde{C}_n + \alpha_2M)\bar{J}_{mn}] + \left[-A_{mn}\frac{d'}{d} + M\left(\frac{\tilde{C}_n}{M}\right)'(1 + MC_m) + \frac{M'}{G}(2C_m\tilde{C}_n + M(1-\gamma)C_m)\right]I_{mn} + D_m\left[M^2\left(\frac{\tilde{D}_n}{M}\right)' + \tilde{D}_n\frac{M'}{G}\right]K_{mn} + [\lambda_n'(\tilde{A}_{mn} + C_m) - M\tilde{D}_nD_m\lambda_m']\bar{J}_{mn} + [\lambda_m'(\tilde{A}_{mn} - \tilde{C}_n) - M\tilde{D}_nD_m\lambda_n']\bar{J}_{mn} + d'\tilde{A}_{mn}[1 + (-1)^{n+m}]$$

$$G_{mn} = i[(\lambda_m\tilde{D}_m - \lambda_nD_n)I_{mn} + (\lambda_mD_n - \lambda_n\tilde{D}_m)K_{mn} - \tilde{D}_m(\alpha_1C_n + \alpha_2M)\bar{J}_{mn}] + \left[-B_{mn}\frac{d'}{d} + M\left(\frac{C_n}{M}\right)'(1 + M\tilde{C}_m) + \frac{M'}{G}(2\tilde{C}_mC_n + M(1-\gamma)\tilde{C}_m\right] \times I_{mn} + \tilde{D}_m\left[M^2\left(\frac{D_n}{M}\right)' + D_n\frac{M'}{G}\right]K_{mn} + [\lambda_n'(B_{mn} + \tilde{C}_m) - MD_n\tilde{D}_m\lambda_m']J_{mn} + [\lambda_m'(B_{mn} - C_n) - MD_n\tilde{D}_m\lambda_n']\bar{J}_{mn} + d'B_{mn}[1 + (-1)^{n+m}]$$

$$\tilde{G}_{mn} = i[(\lambda_m\tilde{D}_m - \lambda_n\tilde{D}_n)I_{mn} + (\lambda_m\tilde{D}_n - \lambda_n\tilde{D}_m)K_{mn} - \tilde{D}_m(\alpha_1\tilde{C}_n + \alpha_2M)\bar{J}_{mn}] + \left[-\tilde{B}_{mn}\frac{d'}{d} + M\left(\frac{\tilde{C}_n}{M}\right)'(1 + M\tilde{C}_m) + \frac{M'}{G}(2\tilde{C}_m\tilde{C}_n + M(1-\gamma)\tilde{C}_m\right] \times I_{mn} + \tilde{D}_m\left[M^2\left(\frac{\tilde{D}_n}{M}\right)' + \tilde{D}_n\frac{M'}{G}\right]K_{mn} + [\lambda_n'(\tilde{B}_{mn} + \tilde{C}_m) - M\tilde{D}_n\tilde{D}_m\lambda_m']J_{mn} + [\lambda_m'(\tilde{B}_{mn} - \tilde{C}_n) - M\tilde{D}_n\tilde{D}_m\lambda_n']\bar{J}_{mn} + d'\tilde{B}_{mn}[1 + (-1)^{n+m}]$$

The fact that $\tilde{Q}_{mn}$ and T_{mn} vanish follows from further algebra using Eqs. (A2) and the definitions of the k_n, κ_n, and λ_n.

Acknowledgment

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